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Dirac Equation with Central Potential: Discrete Spectrum of the Hydrogen Atom in the Robertson–Walker Space–Time

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The formulation of the Dirac equation with electromagnetic field for a general space–time is specialized to the Robertson–Walker metric. For a class of physically meaningful electromagnetic potentials the angular part of the wave function separates as in the free-field case. The scheme is explicitly studied for a Coulomb potential. By using a realistic approximation method one recovers the discrete energy levels of the hydrogen atom in Minkowski space. In case of static space–time, the result is exact for zero curvature, while it is approximate for nonzero curvature. The very good order of accuracy of the result is established by a comparison with similar qualitative and perturbative results.

1. INTRODUCTION

It is well known that the formulation of the Dirac equation can be extended to curved space–time in a general way (Penrose and Rindler, 1990). The procedure is based originally on the idea of the minimal gravitational coupling $\partial_a \rightarrow \nabla_a$ in passing from flat to curved space–time. Such formulations do not suffer from the consistency problem of similar equations relative to higher values of the spin (Penrose and Rindler, 1990; Buchdahl, 1962, 1982; Wünsch, 1985; Illge, 1993).

The electromagnetic field of potential A can be included in the scheme as a generalization of the usual substitution $\partial_a \rightarrow \partial_a - ieA_a$ that follows the canonical quantization. This procedure also can be done in a general space–time (Illge, 1993), while it is not always consistent for values of the spin greater than 1 (Fierz and Pauli, 1939). Besides the study of the general

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formulation, explicit solutions of the (free) Dirac equation in concrete space–time have been obtained. The most representative example is Chandrasekhar's integration of the Dirac equation relative to the Kerr metric (Chandrasekhar, 1983). It is obtained by a separation method by using the Newman–Penrose formalism (Newman and Penrose, 1962). It turns out that the separation method is a simple and powerful tool for the solution of the Dirac equation if, besides the general formal solution (Penrose and Rindler, 1990), one is also interested in practical solutions. With some elementary variants, the separation method can be applied to integrate the Dirac equation also in a time-dependent metric such as the Robertson–Walker (RW) metric (Zecca, 1996).

It is of interest to consider the Dirac equation with explicit electromagnetic potentials in concrete examples of space–time.

In this paper we study the Dirac equation with a Coulomb potential in the case of the RW space–time. The formulation is done by the spinor language of Newman and Penrose. It turns out that for a class of central electromagnetic spinor potentials, the angular part of the wave function can be separated. The resulting angular dependence is the same as in the absence of electromagnetic field. The theory is then applied to the case of a Coulomb potential. Owing to an objective difficulty in separating the r and t dependences the solution is performed in the special case of a constant "radius of the universe." This assumption is very plausible on physical grounds by taking into account that the atomic time intervals are negligible on a cosmological scale.

Under this hypothesis, the radial and time dependences can be separated and integrated by generalizing a standard method. The flat-space–time case gives for the hydrogen atom the same discrete spectrum as in the Minkowski case. Under an approximation (that is very good, as follows by a comparison with related work), the result is independent of the curvature of the space. With respect to the scattering states, the static flat-space–time case recovers the Minkowski situation. In the case of positive and negative space curvature the scattering states have different asymptotic behavior and it is difficult to determine them exactly.

2. THE DIRAC EQUATION WITH ELECTROMAGNETIC POTENTIAL

In the spinor language for curved space–time (Newman and Penrose, 1962) the Dirac equation with electromagnetic field takes the form

$$
\begin{aligned}
(\nabla_{AA'} + iV_{AA'})P^A + i\mu_\star \overline{Q}_{A'} &= 0 \\
(\nabla_{AA'} - i\overline{V}_{AA'})Q^A + i\mu_\star \overline{P}_{A'} &= 0
\end{aligned}
\tag{1}
$$

where $\mu_\star\sqrt{2}$ is the mass of the particle, $\nabla_{AA'}$ is the covariant spinor derivative, and $V_{AA'}$ is the spinor potential of the electromagnetic field. The formulation in Eq. (1) is equivalent to the general form of the spin-1/2 field equation as discussed by Penrose and Rindler (1990). The imaginary unit has been put into evidence in order to compare the present scheme with that used in Zecca (1996), where notations and conventions of Chandrasekhar's book are adopted.

Equation (1) can be developed by using the linking relation $\nabla_{AA'} = \sigma^a_{AA'}\nabla_a$. By making explicit the covariant derivatives in terms of the tabulated spin coefficients (Penrose and Rindler, 1990; Chandrasekhar, 1983) with the identifications $D \equiv \partial_{00'}$, $\delta \equiv \partial_{01'}$, $\delta^\star \equiv \partial_{10'}$, $\Delta \equiv \partial_{11'}$, one finds

$$
\begin{aligned}
(D + \epsilon - \rho + iV_{00})P_1 - (\delta^\star + \pi - \alpha + iV_{10})P_0 + i\mu_\star\bar{Q}_0 &= 0\\
(\delta + \beta - \tau + iV_{01})P_1 - (\Delta + \mu - \gamma + iV_{11})P_0 + i\mu_\star\bar{Q}_1 &= 0\\
(D + \bar{\epsilon} - \bar{\rho} - i\bar{V}_{00})Q_1 - (\delta^\star + \bar{\pi} - \bar{\alpha} - i\bar{V}_{10})Q_0 + i\mu_\star\bar{P}_0 &= 0\\
(\delta + \bar{\beta} - \bar{\tau} - i\bar{V}_{01})Q_1 - (\Delta + \bar{\mu} - \bar{\gamma} - i\bar{V}_{11})Q_0 + i\mu_\star\bar{P}_1 &= 0 \qquad (2)
\end{aligned}
$$

This is the explicit form of Dirac equation in a general curved space–time. Further simplifications require concrete space–time model or symmetry properties of the spin coefficients and of the directional derivatives.

3. CENTRAL POTENTIALS IN THE ROBERTSON–WALKER METRIC

We now confine ourselves to equation (2) in the case of the Robertson–Walker metric of the form

$$
ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - ar^2} + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \qquad (a = 0, \pm 1) \quad (3)
$$

To further study equation (2) by the Newman–Penrose formalism, we choose the null tetrad frame $\{l^i, n^i, m^i, m^{\star i}\}$ previously used in different situations (Zecca, 1996; Montaldi and Zecca, 1994) in terms of which the directional derivatives and the spin coefficients (which we give for the reader's convenience) are given respectively by

$$
\begin{aligned}
D &\equiv l^i\partial_i = 2^{-1/2}[\partial_t + R^{-1}(1 - ar^2)^{1/2}\,\partial_r]\\
\Delta &\equiv n^i\partial_i = 2^{-1/2}[\partial_t - R^{-1}(1 - ar^2)^{1/2}\,\partial_r]\\
\delta &\equiv m^i\partial_i = 2^{-1/2}[\partial_\theta + i\csc\theta\,\partial_\phi]/(rR)\\
\delta^\star &\equiv m^{\star i}\partial_i = 2^{-1/2}[\partial_\theta - i\csc\theta\,\partial_\phi]/(rR) \qquad (4)
\end{aligned}
$$

$$\rho = -2^{-1/2}[r\dot{R} + (1 - ar^2)^{1/2}]/(rR)$$
$$\mu = 2^{-1/2}[r\dot{R} - (1 - ar^2)^{1/2}]/(rR)$$
$$\beta = -\alpha = 2^{-3/2} \cot\theta/(rR), \qquad \varepsilon = -\gamma = 2^{-3/2}\dot{R}/R$$

In view of further simplification it is convenient to set

$$F_1 = P_1, \qquad F_2 = -P_0, \qquad G_1 = -\bar{Q}_0, \qquad G_2 = -\bar{Q}_1 \tag{5}$$

The general form of the Dirac equation with electromagnetic field in the RW space–time is then

$$\begin{aligned}(D + \epsilon - \rho + iV_{00})F_1 + (\delta^\star - \alpha + iV_{10})F_2 &= i\mu_\star G_1\\ (\Delta + \mu - \gamma + iV_{11})F_2 + (\delta - \alpha + iV_{01})F_1 &= i\mu_\star G_2\\ (D + \epsilon - \rho + iV_{00})G_2 - (\delta - \alpha + iV_{10})G_1 &= i\mu_\star F_2\\ (\Delta + \mu - \gamma + iV_{11})G_1 - (\delta^\star - \alpha + iV_{01})G_2 &= i\mu_\star F_1\end{aligned} \tag{6}$$

If the spinor potential is of central type and diagonal, the angular part of the wave function separates as in the free-field case. Suppose indeed

$$V_{AA'} = V_{AA'}(r, t), \qquad V_{01} = V_{10} = 0 \tag{7}$$

Accordingly, the ϕ dependence can be directly separated in equation (6) by the substitution $(F, G) \to (F, G)\exp(im\phi)$, $m = 0, \pm1, \pm2, \ldots$. By defining, as in the free-field case,

$$L^{\pm} = \partial_\theta \mp \frac{m}{\sin\theta} + \frac{1}{2}\cot\theta \tag{8}$$

equation (6) becomes

$$\begin{aligned}rR\sqrt{2}(D + \epsilon - \rho + iV_{00})F_1 + L^- F_2 &= i\mu_\star rR\sqrt{2}G_1\\ rR\sqrt{2}(\Delta + \mu + \epsilon + iV_{11})F_2 + L^+ F_1 &= i\mu_\star rR\sqrt{2}G_2\\ rR\sqrt{2}(D + \epsilon - \rho + iV_{00})G_2 - L^+ G_1 &= i\mu_\star rR\sqrt{2}F_2\\ rR\sqrt{2}(\Delta + \mu + \epsilon + iV_{11})G_1 - L^- G_2 &= i\mu_\star rR\sqrt{2}F_1\end{aligned} \tag{9}$$

where the wave function depends now on r, θ, t. By setting

$$\begin{aligned}rR(t)F_1 &= H_1(r, t)S_1(\theta), \qquad rR(t)F_2 = H_2(r, t)S_2(\theta)\\ rR(t)G_1 &= H_2(r, t)S_1(\theta), \qquad rR(t)G_2 = H_1(r, t)S_2(\theta)\end{aligned} \tag{10}$$

one gets

$$\frac{rR\sqrt{2}}{H_2}[DH_1 + (\epsilon + iV_{00})H_1] - i\mu_\star rR\sqrt{2} = -\frac{L^- S_2}{S_1} = -\lambda$$
$$\frac{rR\sqrt{2}}{H_1}[\Delta H_2 + (\epsilon + iV_{11})H_2] - i\mu_\star rR\sqrt{2} = -\frac{L^+ S_1}{S_2} = \lambda$$
$$\frac{rR\sqrt{2}}{H_2}[DH_1 + (\epsilon + iV_{00})H_1] - i\mu_\star rR\sqrt{2} = \frac{L^+ S_1}{S_2} = -\lambda$$
$$\frac{rR\sqrt{2}}{H_1}[\Delta H_2 + (\epsilon + iV_{11})H_2] - i\mu_\star rR\sqrt{2} = \frac{L^- S_2}{S_1} = \lambda \tag{11}$$

The angular eigenvalue problem originating from the separation of equation (11), $L^- L^+ S_1 = -\lambda^2 S_1$ and $L^+ L^- S_2 = -\lambda^2 S_2$, gives $\lambda^2 = (l + 1)^2$, $(l = 0, 1, 2, 3, \ldots)$ if $m = 0$ and $\lambda^2 = (l + 1/2)^2$ $(l = |m|, |m| + 1, |m| + 2, \ldots)$ if $|m| \geq 1$ (Montaldi and Zecca, 1994). One is then left with

$$DH_1 + (\epsilon + iV_{11})H_1 = \left(i\mu_\star - \frac{\lambda}{rR\sqrt{2}}\right)H_2$$
$$\Delta H_2 + (\epsilon + iV_{00})H_2 = \left(i\mu_\star + \frac{\lambda}{rR\sqrt{2}}\right)H_1 \tag{12}$$

If the spinor potential satisfies the condition

$$V_{00} = V_{11} = V(r, t) \tag{13}$$

by using as independent variables

$$\tau = \tau(t) = \int_0^t \frac{dt}{R(t)}, \qquad s = s\,(r) = \int_0^r \frac{dr}{\sqrt{1 - ar^2}} \tag{14}$$

so that $D = (\partial_\tau + \partial_s)/(\sqrt{2}R)$, $\Delta = (\partial_\tau - \partial_s)/(\sqrt{2}R)$, and by further setting

$$rf = H_1 - H_2, \qquad rg = H_1 + H_2 \tag{15}$$

equation (12) becomes, in terms of the functions f and g,

$$f_s + g_t + \frac{r_s - \lambda}{r} f + g(2^{-1}\,\dot{R} - i\mu_\star\sqrt{2}R + i\sqrt{2}RV) = 0$$
$$g_s + f_\tau + \frac{r_s + \lambda}{r} g + f(2^{-1}\,\dot{R} + i\mu_\star\sqrt{2}R + i\sqrt{2}RV) = 0 \tag{16}$$

where $f_s = \partial f/\partial s$, $\dot{R} = dR/dt$, and $r_s = \sqrt{1 - ar^2}$.

4. HYDROGEN ATOM

It is not difficult to show that equation (16) does not admit a solution of the form $f = A(s)T(\tau)$, $g = B(s)S(\tau)$. [One can reach the same conclusion by applying the reduction method developed for the free-field case (Zecca, 1996).] It is possible to study equation (16) under the conditions

$$ar^2 \ll 1 \qquad (r_s \approx 1)$$
$$R = \text{const} = R_0 \tag{17}$$

R_0 is the radius of the universe at the present time. The assumptions (17) are very plausible for the study of the bounded states of the hydrogen atom. The atomic dimensions and the time intervals involved are negligible on a cosmological scale to a very good approximation.

We now apply the previous scheme to the case of a Coulomb potential $A_\mu \cong (1/r, 0, 0, 0)$. The corresponding spinor potential is, by a suitable choice of the proportionality constant and by using the explicit form of the assumed null tetrad frame,

$$V_{AA'} = \sigma^t_{AA'} A_t = \frac{1}{\sqrt{2}\, R_0} \frac{\chi}{r} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{18}$$

As a consequence of equations (17) the τ dependence factors out in the form $\exp(ikR_0\tau) = \exp(ikt)$ and equations (16) become

$$f' + \frac{1-\lambda}{r} f + ig\left(k_0 - m_0 + \frac{\chi}{r}\right) = 0$$
$$g' + \frac{1+\lambda}{r} g + if\left(k_0 + m_0 + \frac{\chi}{r}\right) = 0 \tag{19}$$

where $m_0 = \mu_\star \sqrt{2} R_0$, $' = d/ds$, $k_0 = kR_0$, and now f, g depend only on s.

Equation (19) has the formal structure of the radial Dirac equation for the hydrogen atom in Minkowski space and reduces to it when $a = 0$ $(r = s)$ (Berestetski *et al.*, 1972). All cases $a = 0, \pm 1$ can be studied in a uniform way by generalizing a standard procedure of integration (Darwin, 1928; Gordon, 1928) by setting

$$f = Ae^{-\rho/2}\rho^{\gamma-1}\,(q_1 + q_2), \qquad \rho = 2\delta r(s)$$
$$g = Be^{-\rho/2}\rho^{\gamma-1}\,(q_1 - q_2)$$
$$\frac{A}{B} = \left(\frac{k-m}{k+m}\right)^{1/2}, \qquad \delta = R_0(m^2 - k^2)^{1/2} \tag{20}$$

From equations (20) and (19) one gets exactly

$$\rho\sqrt{1-ar^2}q_1' + q_1\left[(\gamma-1)\sqrt{1-ar^2}+1-\frac{\rho}{2}(\sqrt{1-ar^2}-1)\right.$$
$$\left.+\frac{ik\chi}{\sqrt{k^2-m^2}}\right] - q_2\left(\lambda+\frac{i\chi m}{\sqrt{k^2-m^2}}\right)=0$$
$$\rho\sqrt{1-ar^2}\,q_2' + q_2\left[(\gamma-1)\sqrt{1-ar^2}+1-\frac{\rho}{2}(\sqrt{1-ar^2}+1)\right.$$
$$\left.-\frac{i\chi k}{\sqrt{k^2-m^2}}\right] + q_1\left(-\lambda+\frac{i\chi m}{\sqrt{k^2-m^2}}\right)=0 \qquad (21)$$

where $' = d/d\rho$.

It is now possible to study the bounded states of the hydrogen atom by applying once again the approximation $ar^2 \ll 1$ because the corresponding solutions of equations (21) must be localized into atomic dimensions. Equations (21) become then

$$\rho q_1' + \left(\gamma+\frac{i\chi k}{\sqrt{k^2-m^2}}\right)q_1 - \left(\lambda+\frac{i\chi m}{\sqrt{k^2-m^2}}\right)q_2 = 0$$
$$\rho q_2' + \left(-\lambda+\frac{i\chi m}{\sqrt{k^2-m^2}}\right)q_1 + \left(\gamma-\frac{i\chi k}{\sqrt{k^2-m^2}}-\rho\right)q_2 = 0 \qquad (22)$$

By choosing

$$\gamma = (\lambda^2-\chi^2)^{1/2} \qquad (23)$$

we find that the system (22) separates into confluent hypergeometric equations:

$$\rho q_1'' + (1+2\gamma-\rho)q_1' - \left(\gamma-\frac{\chi k}{\sqrt{m^2-k^2}}\right)q_1 = 0$$
$$\rho q_2'' + (1+2\gamma-\rho)q_2' - \left(1+\gamma-\frac{\chi k}{\sqrt{m^2-k^2}}\right)q_2 = 0 \qquad (24)$$

Bounded solutions for $\rho = 0$ are then

$$q_1 = CF\left(\gamma-\frac{\chi k}{\sqrt{m^2-k^2}},\,1+2\gamma;\,\rho\right)$$
$$q_2 = DF\left(1+\gamma-\frac{\chi k}{\sqrt{m^2-k^2}},\,1+2\gamma;\,\rho\right) \qquad (25)$$

$F(a, b; x)$ is the confluent hypergeometric function. The constants C, D are not independent. Their relation comes from (22), (25) with $\rho = 0$:

$$D = \frac{\gamma - \chi k/\sqrt{m^2 - k^2}}{\lambda - \chi m/\sqrt{m^2 - k^2}} C \tag{26}$$

The discrete spectrum of the hydrogen atom corresponds to the condition on the wave function (cf. Zecca, 1995; Montaldi and Zecca, 1996)

$$\int d_3x \sqrt{-g}(|P^0|^2 + |P^1|^2 + |Q^1|^2 + |Q^0|^2) < \infty \tag{27}$$

which finally implies, by the previous substitutions and by factoring out the (bounded) angular integral,

$$\int_0^\infty d\rho\, \rho^{2\gamma} e^{-\rho}(|q_1|^2 + |q_2|^2) < \infty \tag{28}$$

By taking into account that the asymptotic behavior of the hypergeometric function is $\sim\exp(\rho)$ (Abramovitz and Stegun, 1970), we find that the convergence of the integral (28) requires the truncation of both the hypergeometric series

$$\gamma - \frac{\chi k}{\sqrt{m^2 - k^2}} = -n_r, \qquad n_r = 1, 2, 3, \ldots \tag{29}$$

[If $n_r = 0$, then, from (22), $\chi m/\sqrt{m^2 - k^2} = \pm\lambda$. Therefore if $\lambda < 0$, then $D = 0$, by equation (20), and the solution is still acceptable (Berestetski *et al.*, 1972).] The discrete energy levels are then

$$\frac{k}{m} = \left[1 + \frac{\chi^2}{(n_r + \sqrt{\lambda^2 - \chi^2})^2}\right]^{-1/2} \tag{30}$$

The result (30) is independent of the value of $a = 0, \pm 1$ and coincides with the Minkowski-space situation (Bethe and Salpeter, 1957). Under the second of conditions (17) the result is exact for $a = 0$, while it represents a good approximation for the curved space–time cases.

With respect to the scattering states, the static case $a = 0$ of equation (21) coincides with the Minkowski-space case and its solution is well known (e.g., Berestetski *et al.*, 1972; Bethe and Salpeter, 1957).

The asymptotic behavior for large r of the solutions of equations (16), (21) for $a = -1$ is essentially different than for $a = 0$. In the curved space cases an explicit exact solution is difficult to determine due to the presence of the square root terms, which cannot be neglected.

5. COMPARISON WITH SIMILAR RESULTS

The results relative to the spectrum of the hydrogen atom are in a complete agreement with the general results of Audretsch and Schäfer (1978a, b) as well as with the perturbative calculations of Parker (1980a, b). The quantitative results obtained by Parker concern the correction of the Minkowski spectrum (30) produced by the local curvature in a given general space–time. In order to produce an energy-level shift of the same order of magnitude as the Lamb shift of the hydrogen atom, one must have at least $D \approx 2 \times 10^{-3}$ cm, where D is the characteristic radius of curvature of the space–time at the location of the hydrogen atom: $D^{-2} \sim R_{\alpha\beta\gamma\delta}$, with $R_{\alpha\beta\gamma\delta}$ being the Riemann tensor. In our case, with the notation of the previous section, $D \sim 1/R_0^2 \ll 2 \times 10^{-3}$ cm. This establishes the accuracy of our approximation.

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